In the analysis, the propagation characteristics of the wave are determined by formulating an eigenvalue problem, where its eigenvalues are the propagation constants squared and the eigenvectors represent directly the expansion coefficients of the propagating fields. The analysis treats the inhomogeneity as a polarization current exciting the corresponding homogeneous guiding system.

The method is used to solve the problem of electromagnetic wave propagation in partially-filled parallel-plate waveguide. The convergence, dispersion characteristics, and accuracy of the method are studied and compared to the results of the variational method [6]. The present method proved to have better accuracy than the variational method [6].

REFERENCES

[1] A. S. Omar and K. F. Schüneman, "Complex and backward-wave modes in inhomogeneously and anisotropically filled waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, no. 3, pp. 268–275, Mar. 1987.

[2] A. S. Omar and K. F. Schüenmann, "Application of the generalized spectral-domain technique to the analysis of rectangular waveguides with rectangular and circular inserts," *IEEE Trans. Microwave Theory Tech.*, vol. 39, no. 6, pp. 944–952, June 1991.

[3] M. Mohamed and J. Jarem, "A multipole analysis of a dielectric loaded coaxial rectangular waveguide," *IEEE Trans. Microwave Theory Tech.*, vol. 40, no. 3, pp. 482–494, Mar. 1992.

[4] R. F. Harrington, *Time-Harmonic Electromagnetic Fields.* New York: McGraw-Hill, 1961.

[5] D. M. Pozar, *Microwave Engineering.* New York: Addison-Wesley, 1990.

[6] M. Z. Mohamed and J. M. Jarem, "Variational analysis of inhomogeneous parallel-plate waveguide modes," in *1991 Int. Simulation Technology Conf., SimTec'91,* Oct. 21–23, 1991, Orlando, FL, pp. 145–150.

An Efficient Numerical Procedure Using the Shifted Power Method for Analyzing Dielectric Waveguides Without Inverting Matrices

Ching-Chuan Su

***Abstract*— A numerical procedure using the finite-difference scheme and the shifted power method is used to analyze the propagation characteristics of dielectric waveguides. The unique feature of this procedure is that in determining the eigenvalues corresponding to dominant modes no operation as costly as matrix inversion, such as Gaussian elimination, LU decomposition, or tridiagonalization, is invoked. So the proposed procedure is rather efficient in both memory space and computer time. Numerical results of a circular step-index fiber are presented for comparison. Due to its efficiency, the proposed procedure is capable of analyzing coupled waveguides.**

I. INTRODUCTION

Applying the finite-difference or finite-element method to analyzing the propagation characteristics of dielectric waveguides has been investigated for a long time and by many people. These numerical methods render the partial differential equation governing the propagation characteristics of dielectric waveguides to linear simultaneous algebraic equations, which are manageable on a computer. In essence the matrix equation is an eigenvalue problem. Depending on the mathematical formulation, the properties of boundary conditions, and the methods of evaluating the eigenvalues, the eigenvalue problem can be written in several forms, which will be discussed in detail in Section II.

In waveguide theory the important eigenvalues corresponding to dominant guided modes in which one is interested are the largest propagation constants or the lowest frequencies. To calculate these particular eigenvalues several methods have been employed, such as the zero-determinant searching, the inverse power method [1, ch. 10], the subspace iteration method [2] (a variation of the inverse power method using simultaneous iteration), and the method involving tridiagonalization and Sturm sequence [1, chs. 8 and 9]. For evaluating the determinant of a matrix or for inverting a matrix in using the inverse power method one needs to use, for example, Gaussian elimination or LU decomposition. The matrix resulting from applying the finite-difference or finite-element method to a differential equation is banded. The bandwidth grows with the dimension of the problem. For one-, two-, and three-dimensional cases, the value of bandwidth M are of order unity, $O(N^{1/2})$, and $O(N^{2/3})$, respectively, where N is the matrix order. For a band matrix the number of required operations for matrix inversion is of order $O(M^2N)$. The number of required operations for tridiagonalizing a symmetric matrix is of the same order as that of matrix inversion. Thus for the two-dimensional waveguide problem the required computation will go as N^2, when conventional approaches were used.

After analyzing the distribution of eigenvalues of an associated matrix in Section III, we show that the direct power method can be used to calculate the eigenvalues and associated eigenfunctions corresponding to dominant modes by suitably shifting the eigenvalues. The unique feature of this procedure is that no matrix inversion or tridiagonalization is invoked. Thus the proposed procedure is efficient in both computation speed and memory space and is simple in the programming work. A major drawback of the proposed procedure is the slow convergence rate. Some methods to accelerate this rate will be discussed in Section V. Numerical results for circular step-index fiber and coupled rectangular waveguides are presented in Section VI.

II. EIGENVALUE PROBLEMS

Consider a transversely inhomogeneous dielectric waveguide in which a transverse field ψ satisfies the scalar wave equation

$$\nabla_t^2\psi(x,y) + [k_0^2\epsilon(x,y) - \beta^2]\psi(x,y) = 0, \tag{1}$$

where $k_0^2 = \omega^2\mu_0\epsilon_0$, $\epsilon(x,y)$ denotes relative permittivity distribution, and β is the propagation constant in the axial direction. Suppose that dielectric waveguide is cladded by a homogeneous medium with relative permittivity ϵ_1 and the maximum value of the permittivity $\epsilon(x,y)$ is ϵ_2. It is of convenience to express the permittivity $\epsilon(x,y)$ as

$$\epsilon(x,y) = \epsilon_1 + (\epsilon_2 - \epsilon_1)P(x,y), \tag{2}$$

where the profile $P(x,y)$ is zero in the cladding and its maximum value is unity. Using the normalized propagation constant B and normalized frequency V:

$$B = \frac{(\beta/k_0)^2 - \epsilon_1}{\epsilon_2 - \epsilon_1} \tag{3}$$

$$V = k_0 b\sqrt{\epsilon_2 - \epsilon_1}/\pi, \tag{4}$$

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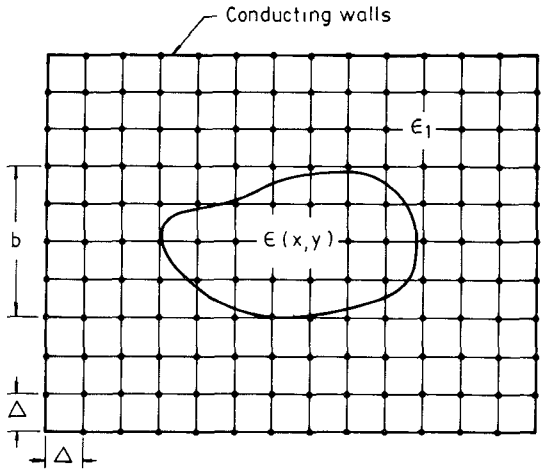

Fig. 1. Cross section of an arbitrarily-shaped waveguide, the domain of calculation, and the rectangular grid for finite-difference scheme.

the wave equation becomes

$$\nabla_t^2\psi(x,y) + (\pi V/b)^2 P(x,y)\psi(x,y) = (\pi V/b)^2 B\psi(x,y), \quad (5)$$

where b denotes the side length of a rectangle enclosing the cross section of waveguide's core region. It is known that the value of normalized propagation constant B ranges from zero to unity and practical value of normalized frequency V ranges from zero to a quantity of order unity.

By a use of the finite-difference technique, the partial differential equation becomes the linear algebraic simultaneous equations:

$$\begin{aligned}\{\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{ij}\} \\ + (\pi V\Delta/b)^2 P_{ij}\psi_{ij} \qquad i = 1,2\cdots m_1 \\ = (\pi V\Delta/b)^{2B}\psi_{ij}, \qquad j = 1,2\cdots m_2 \end{aligned} \quad (6)$$

where $\psi_{ij} = \psi(i\Delta, j\Delta), P_{ij} = P(i\Delta, j\Delta)$, and Δ is a small increment in length. The increment Δ should be much less than the side length b in order to model the waveguide and the modal fields accurately. The rectangular grid of finite-difference scheme is depicted in Fig. 1. Since the modal fields penetrate into the cladding, the domain of calculation is extended out from the core region. On the boundary of the domain, a boundary condition of vanishing field is enforced ($\psi_{ij} = 0$ for $i = 0, m_1 + 1$, or $j = 0, m_2 + 1$). Physically, such a boundary condition corresponds to shielding the waveguide with a (two-dimensional) rectangular cavity of metallic walls. In order to conform to the finite-difference scheme a rectangular domain of size $(m_1+1)\Delta$ by $(m_2+1)\Delta$ is used, regardless of the waveguide's structure. The extent to which the modal field penetrates into the cladding depends on several factors, especially the constant B. Whether the extension of domain is large enough can be monitored just by checking whether the calculated fields at the node points neighboring to the boundary are small enough.

To solve the simultaneous equations three approaches are commonly used. The first one is to arrange the simultaneous equations in the form:

$$\boldsymbol{R}\boldsymbol{x} = 0, \quad (7)$$

where $\boldsymbol{x}$ is a vector whose elements are the various field values ψ_{ij} and $\boldsymbol{R}$ is a matrix whose elements depend on the eigenvalue β (or ω). The eigenvalue problem can be solved by searching a suitable value of β (or ω) such that the determinant of matrix $\boldsymbol{R}$ vanishes, as used in [3]. For evaluating the determinant a process as costly as matrix inversion must be invoked. This approach needs to calculate many determinants and is less efficient, but is capable of dealing with sophisticated boundary conditions.

The second approach is to fix the value of B and arrange the simultaneous equations in the form of generalized eigenvalue problem:

$$\boldsymbol{S}\boldsymbol{x} = \lambda\boldsymbol{T}\boldsymbol{x}, \quad (8)$$

where matrix $\boldsymbol{S}$ comes from the first term on the left-hand side of (6), matrix $\boldsymbol{T}$ comes from the other two terms in (6), and the eigenvalue $\lambda = V^2$. In this form the important eigenvalues V are the smallest ones, which can be found using the inverse power method or the subspace iteration method, as used in [4]. This approach also requires one to invert or, equivalently, factorize the matrix $\boldsymbol{S}$.

The third approach is to fix the value of V and arrange the simultaneous equations in the form of standard eigenvalue problem:

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}, \quad (9)$$

where matrix $\boldsymbol{A}$ comes from the two terms on the left-hand side of (6) and eigenvalue $\lambda = (\pi V\Delta/b)^2 B$. Since the important eigenvalues B are of the largest ones, it seems that these eigenvalues can be found quite efficiently by using the ordinary power method and its variations in which no matrix inversions are involved. However, this approach does not work. The trouble is that the calculated eigenvalues do not correspond to guided modes and hence are spurious. Instead, more expensive methods such as the subspace iteration method [5] and the methods involving tridiagonalization [6] have been used to solve standard eigenvalue problems similar to (9).

Thus all the three popular approaches involve a process similar to matrix inversion with respect to computational cost and hence are inefficient in both memory space and computer time. The origin of the just-mentioned spurious eigenvalues can be given after analyzing the distribution of the eigenvalues of matrix $\boldsymbol{A}$, as discussed in the following section.

III. Distribution of Eigenvalues

For simplicity in the discussion we first remove the waveguide, that is, put $P(x,y) = 0$. Note that matrix $\boldsymbol{A}$ is real and symmetric. Thus all the eigenvalues λ of matrix $\boldsymbol{A}$ are real. An actual calculation using the power method and its variation yields that the largest-magnitude eigenvalue λ is close to -8 and the smallest-magnitude one is also negative and close to zero. Thus all the eigenvalues λ are distributed between zero and -8, which agrees with Gerschgorin's theorem [1, ch. 1]. Physically, these negative eigenvalues correspond to resonance frequencies of the cavity resonator formed by the artificially imposed metallic boundary.

The introduction of a dielectric waveguide will modify some of the diagonal elements of matrix $\boldsymbol{A}$. At practical values of frequency V the modification is just slight, since the quantity $(\pi V\Delta/b)^2$ is much less than unity, unless at very large V. Accordingly, the eigenvalues are expected to be modified slightly. Some positive eigenvalues λ will emerge, which, when divided by the small quantity $(\pi V\Delta/b)^2$, become the normalized propagation constants B of guided modes. Since the constant B is less than unity, these positive eigenvalues λ are much less than unity. An actual calculation shows that the largest-magnitude eigenvalue λ is still close to -8, which is much larger in magnitude than the eigenvalues λ corresponding to guided modes. This is why the ordinary power method yields spurious eigenvalues and hence can not be used to find the eigenvalues λ corresponding to guided modes. On knowing the origin of these spurious eigenvalues, the trouble can be removed by using the shifted power method, as discussed in the following section.

IV. Shifted Power Method

In the shifted power method the eigenvalue problem (9) is modified as:

$$(\boldsymbol{A}+s\boldsymbol{I})\boldsymbol{x}=(\lambda+s)\boldsymbol{x}, \\ =\zeta\boldsymbol{x} \tag{10}$$

where $\boldsymbol{I}$ is the unit matrix and s is a suitable constant. If λ is an eigenvalue of matrix $\boldsymbol{A}$, ζ is an eigenvalue of matrix $(\boldsymbol{A}+s\boldsymbol{I})$ with the same eigenvector. The key point of the proposed procedure is that *let s be, say, 8, then the largest-magnitude eigenvalues ζ will correspond to the propagation constants of dominant modes.* Since matrix $(\boldsymbol{A}+s\boldsymbol{I})$ is still symmetric, the eigenvalues of largest magnitude and the associated eigenfunctions can be found using the block power method (or called treppen iteration) or the method of lopsided iteration [1, ch. 10], which are variations of the power method employing simultaneous iteration and orthogonalization.

The proposed procedure is efficient, since the major computation is just the multiplications of matrix $(\boldsymbol{A}+s\boldsymbol{I})$ with various vectors $\boldsymbol{x}$ or, actually, the algebraic operations:

$$\psi_{i+1,j}+\psi_{i-1,j}+\psi_{i,j+1}+\psi_{i,j-1} \\ +[(\pi V\Delta/b)^2P_{ij}-4+s]\psi_{ij}.$$

It is important to note that both the total numbers of required algebraic operations and memory space are *linearly* proportional to $N(=m_1\times m_2)$, the total number of unknowns. A similar method has been used to calculate dominant resonant modes in cavities [7].

V. Convergence Rate

The convergence rate is determined by the starting vectors and how fast the term (ζ_{q+1}/ζ_p) goes to zero [1, ch. 10], where ζ_i denotes the ith eigenvalue, p is the number of modes interested, q is the number of vectors employed in the simultaneous iteration, and n is the number of iterations. A major drawback of the proposed method is that the convergence rate is slow. This is because that the ratio ζ_{q+1}/ζ_p may be quite close to unity. The reason is twofold. First, since the number of eigenvalues of matrix $\boldsymbol{A}$ is equal to the number of unknowns, which can be as large as several hundred or more, so the eigenvalues ζ are closely packed. Secondly, if the eigenvalues are shifted by an amount s which is much greater than λ_p and λ_{q+1}, then the ratio will be close to unity, as shown by the relation:

$$\zeta_{q+1}/\zeta_p\simeq 1-(\lambda_P-\lambda_{q+1})/s. \tag{11}$$

The second effect can be remedied somewhat by choosing a constant s less than 8. It is seen that a constant s is suitable so long as it is greater than one half of the absolute value of the largest-magnitude eigenvalue λ, which is close to 4. The first effect is discussed in detail in the following paragraph.

Since the eigenvalue $\lambda=(\pi V\Delta/b)^2B$, the ratio depends on the frequency V and the increment Δ. For a given waveguide structure a smaller Δ increases the accuracy, but increases the computation twofold. First, a smaller Δ increases the total number of unknowns and hence that of algebraic operations for each iteration. Secondly, a smaller Δ decreases the magnitudes of eignevalues λ corresponding to dominant modes and thus slows the convergence rate. To calculate the propagation constant of the fundamental mode near cutoff (at a very low value of V) may have some difficulty, since the corresponding modal fields penetrate very deeply into the cladding and hence one should sufficiently extend the domain of calculation. Furthermore, like a smaller Δ, a lower V also decreases the magnitudes of eigenvalue λ and hence slows the convergence rate. Fortunately, since the corresponding field variation becomes small at a lower V, the value of Δ can be increased to reduce the number of unknowns and accelerates the convergence rate in compensation. In constructing the β-ω (B-V) dispersion curves, where the value of V is varied step by step, the convergence rate can be improved, if we use the eigenfunctions calculated at the previous value of V as the starting vectors.

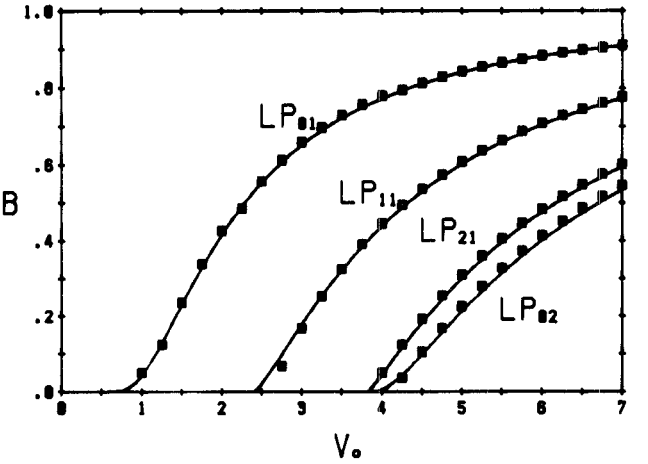

Fig. 2. Dispersion curves of the first four scalar modes in a step-index fiber. The normalized frequency used here is given as $V_0=k_0a\sqrt{\epsilon_2-\epsilon_1}$, where a is the fiber's radius.

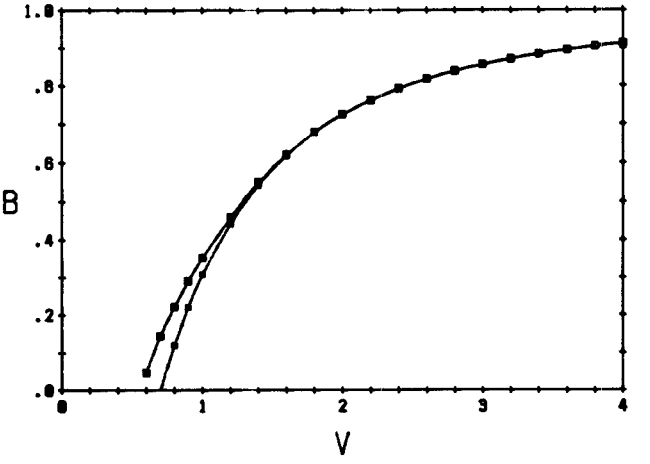

Fig. 3. Dispersion curves of the first two modes of two coupled identical square waveguides.

VI. Numerical Results

In this section we present the calculated results for a circular step-index fiber and coupled rectangular waveguides. The simple staircase approximation is used to model the cross sections of all the waveguides.

We use a 20×20 rectangular grid to model the cross section of a circular step-index fiber ($\Delta/b=1/20$), except for the LP_{01} mode at $V_0<4$. (The normalized frequency used for circular fiber is given conventionally as $V_0=k_0a\sqrt{\epsilon_2-\epsilon_1}$, where a is the fiber's radius.) While, 10×10 and 6×6 grids are used to handle the LP_{01} mode at $1<V_0<4$ and $V_0=1$, respectively. The numbers of unknowns are near 1500. The numbers of iteration n to reach convergence at lower V_0 are around several hundred ($n=800$ at $V_0=1$), and those at higher V_0 are around one hundred, when the starting vectors were chosen randomly and the parameters p and q in (11) are equal. The

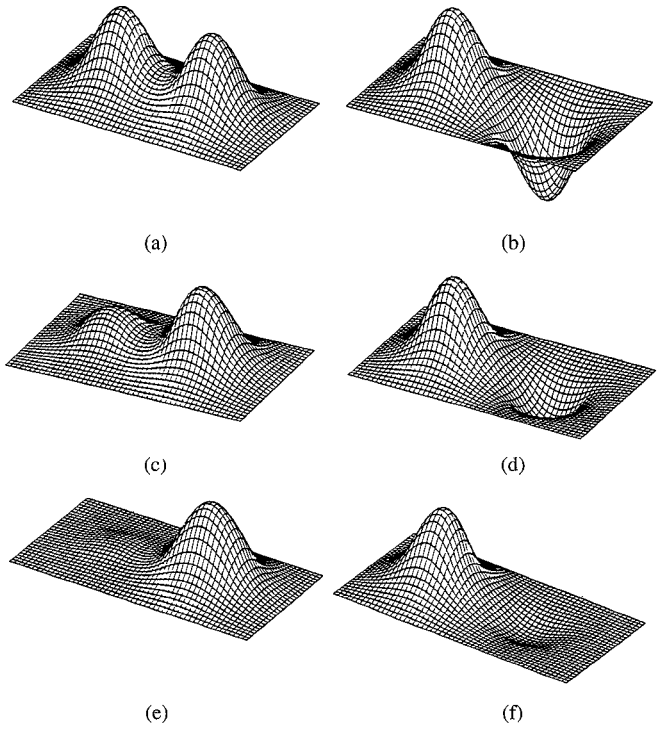

Fig. 4. Fishnet plots of the field distributions of the first two modes in two coupled rectangular waveguides separated by a distance b at $V = 1$. The sizes of the two cross sections are $b \times b$ and $c \times b$. The aspect ratio $c/b = 1.0$ for (a) and (b), 1.1 for (c) and (d), and 1.4 for (e) and (f).

calculated results are presented in the squares in Fig. 2. Compared with the exact solutions (solid lines) it is seen that the accuracy of the calculation is rather high. This is because that the efficiency of this method permits us to use a very fine discretization.

The proposed method is also capable of dealing with coupled waveguides by treating the whole coupled waveguides as a single waveguide. The dispersion relations of the first two modes of two coupled square waveguides of side length b and separated by a distance b are presented in Fig. 3. The corresponding modal fields over the calculation domain (of size $5b$ by $3b$) at normalized frequency $V = 1$ are shown in Figs. 4(a) and (b). Since the two waveguides are identical, one of the modal fields is symmetric and the other is antisymmetric. Such a symmetry is deteriorated by making the two waveguides non-identical. To see this, we replace one of the square waveguides with a rectangular one of cross section $c \times b$. As seen from Fig. 4, one of two peaks in each field distribution diminishes gradually when the side length c is increased. This calculation thus supports that the coupling between two non-identical waveguides is weak, as predicted in the coupled-mode theory. The numbers of iteration n in obtaining the results in Fig. 4, where the starting vectors are chosen randomly, are around 250 (with $V = 1, \Delta/b = 0.1$, and $N \sim 1500$).

VII. Conclusion

A numerical procedure using the finite-difference method has been used to analyze propagation characteristics of dielectric waveguides. The unique feature of this procedure is that the eigenvalues and associated eigenfunctions are solved iteratively using the shifted power method in which no matrix inversion or the equivalent is invoked. Both the requirements of memory storage space and computation time are linearly proportional to N, the total number of unknowns. Thus the proposed procedure is quite efficient and the problem with N as large as a few thousand can be handled on a personal computer.

References

[1] A. Jennings, *Matrix Computation for Engineers and Scientists*. London: Wiley, 1977.
[2] K. J. Bathe and E. L. Wilson, *Numerical Methods in Finite Element Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1976, sect. 12.3.
[3] K. Oyamada and T. Okoshi, "Two-dimensional finite-element calculation of propagation characteristics of axially nonsymmetrical optical fibers," *Radio Science*, vol. 17, pp. 109–116, Jan. 1982.
[4] C. Yeh, S. B. Dong, and W. Oliver, "Arbitrarily shaped inhomogeneous optical fiber or integrated optical waveguides," *J. Appl. Phys.*, vol. 46, pp. 2125–2129, May 1975.
[5] J. Katz, "Novel solution of 2-D waveguides using the finite element method," *Appl Opt.*, vol. 21, pp. 2747–2750, Aug. 1982.
[6] E. Schweig and W. B. Bridge, "Computer analysis of dielectric waveguides: A finite-difference method," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 531–541, May 1984.
[7] T. Weiland, "On the computation of resonant modes in cylindrically symmetric cavities," *Nuclear Instruments and Methods*, vol. 216, pp. 329–348, 1983.